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TWISTED SECOND COHOMOLOGY GROUP OF A FINITELY PRESENTED GROUP

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Abstract: For a finitely presented group G and G -module M , using combinatorial group theory, a new calculation of a twisted second cohomology group $H^2(G, M)$ is introduced. We apply our method to some well-known groups and calculate their second cohomology groups.

Keywords: twisted second cohomology group

1. INTRODUCTION

For a finitely presented group $G = \langle X | S \rangle$, let F be a free group on X and R the normal closure of S in F . If we regard \mathbb{Z} as a trivial G -module, then we have the second homology group

$$H_2(G, \mathbb{Z}) \simeq (R \cap [F, F]) / [F, R]$$

of G by Hopf's formula. (See [2].) On the other hand, if G acts on M non-trivially, then a computation of twisted second (co)homology group $H^2(G, M)$ is much more complicated. In this paper, for a finitely presented group G and a G -module M , we introduce one of methods of a calculation of the second cohomology group $H^2(G, M)$ using combinatorial group theory. Furthermore, we apply our method to some well-known groups, for example, the dihedral group D_n , the special linear group $SL(2, \mathbb{Z})$ and the braid group B_3 of index three.

In this paper, we use the following notation. Let G be a group and M a G -module. We denote the group ring of G over \mathbb{Z} by $\mathbb{Z}[G]$. For any $\alpha \in \mathbb{Z}[G]$, we put

$$M^\alpha = \{m \in M \mid \alpha \cdot m = m\},$$

$$\alpha M = \{\alpha \cdot m \in M \mid m \in M\},$$

where $\alpha \cdot m$ denotes the action of α on m .

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2. THE REIDEMEISTER-SCHREIER METHOD

In this section, we review the Reidemeister-Schreier method. This is one of methods to obtain a presentation for a subgroup H of a given presented group $G = \langle X | S \rangle$. We use the Reidemeister-Schreier method to calculate the second cohomology groups in later sections.

Let F be the free group on X and K a subgroup of F . A subset $T \subset F$ is called Schreier transversal for K in F if T satisfies the following properties

- (1) T is a right coset representative system for K in F ,
- (2) $1 \in T$, where 1 is the identity element of F ,
- (3) (Schreier property) T contains all initial segments of all elements of T , that is,

$$t = x_{\mu_1}^{e_1} x_{\mu_2}^{e_2} \cdots x_{\mu_n}^{e_n} \in T \Rightarrow x_{\mu_1}^{e_1} x_{\mu_2}^{e_2} \cdots x_{\mu_{n-1}}^{e_{n-1}} \in T$$

where $t = x_{\mu_1}^{e_1} x_{\mu_2}^{e_2} \cdots x_{\mu_n}^{e_n}$ is a reduced word and $e_i \in \{\pm 1\}$, ($1 \leq i \leq n$).

Let H be a subgroup of G and H' the inverse image of H under the natural homomorphism $\varphi : F \rightarrow G$. We denote a Schreier transversal for H' in F by T . For any $w \in F$, we define $\bar{w} \in T$ by the rule $H'w = H'\bar{w}$. A map

$$^- : F \rightarrow T \quad w \mapsto \bar{w}$$

is called a right coset representative function for F modulo H' . For any $t \in T$ and $x \in X$ we put

$$(t, x) := tx(\bar{tx})^{-1}, \quad (t, x^{-1}) := (\overline{tx^{-1}}, x)^{-1} \in H'.$$

Let $X^{-1} = \{x^{-1} \mid x \in X\}$. For any word $w = y_1 y_2 \cdots y_n \in F$, $y_i \in X \cup X^{-1}$, we put

$$\tau(w) := (1, y_1)(\overline{y_1}, y_2) \cdots (\overline{y_1 \cdots y_{i-1}}, y_i) \cdots (\overline{y_1 \cdots y_{n-1}}, y_n).$$

The map τ is called the Reidemeister-Schreier rewriting process for H' .

Proposition 2.1. *With the above notation, if we put*

$$\begin{aligned} X' &= \{(t, x) \in H' \mid t \in T, x \in X, (t, x) \neq 1\}, \\ S' &= \{\tau(tst^{-1}) \in H' \mid t \in T, s \in S\}, \end{aligned}$$

then we have

- (1) H' is the free group on X' ,
- (2) $\ker(\varphi|_{H'})$ is the normal closure of S' in H' .

Hence, H has a presentation $H = \langle X' \mid S' \rangle$.

This proposition is well-known fact. For details, see [4].

3. A CALCULATION OF THE SECOND COHOMOLOGY OF A FINITELY PRESENTED GROUP

Let G be a group and M a G -module. We assume that G has a finite presentation $G = \langle X | S \rangle$. Let F be the free group on X , R the normal closure of S in F and T a Schreier transversal for R in F . From the spectral sequence of the group extension

$$1 \rightarrow R \rightarrow F \rightarrow G \rightarrow 1,$$

we have an exact sequence

$$0 \rightarrow H^1(G, M) \rightarrow H^1(F, M) \xrightarrow{\text{res}} H^1(R, M)^G \rightarrow H^2(G, M) \rightarrow H^2(F, M).$$

Since F is the free group, $H^2(F, M) = 0$. Hence, to calculate $H^2(G, M)$, it suffices to calculate the group $H^1(R, M)^G$.

Now, R is a free group. If we can obtain a free basis X' of R , then we can determine a basis of $H^1(R, M)$ as a free abelian group. Furthermore, we see that

$$\begin{aligned} H^1(R, M)^G \\ = \{f \in H^1(R, M) \mid f(\sigma^{-1}x'\sigma) = f(x'), \forall \sigma \in X, \forall x' \in X'\}. \end{aligned}$$

In this paper, to obtain a free basis X' of R , we use the Reidemeister-Schreier method. Then, considering the restriction map $\text{res} : H^1(F, M) \rightarrow H^1(R, M)^G$, we obtain $H^2(G, M)$.

In this method, it is important to construct a Schreier transversal for R in F . The difficulty of the construction of a Schreier transversal depends on not only a given group G but also a presentation for the group G . Hence it is necessary to find a suitable presentation for G .

4. THE CYCLIC GROUP C_n

It is well-known that the (co)homology groups of the cyclic group are completely determined. We, however, dare to apply our method in this case. It is the best way to use a simple example to understand our method. Let C_n be a cyclic group of degree $n \geq 2$. The group C_n has a finite presentation

$$C_n = \langle x \mid x^n = 1 \rangle.$$

Let F be the free group on $\{x\}$ and R the normal closure of $\{x^n\}$ in F .

Lemma 4.1. *The group R is a free group with basis $\{x^n\}$.*

Proof. Since F is an abelian group, it is clear that $\{x^n\}$ is a free basis of R . However, to understand our method, we apply the Reidemeister-Schreier method to this case.

First, we see that $T = \{1, x, \dots, x^{n-1}\}$ is a Schreier transversal for R in F . Hence, a free basis

$$X^* = \{(t, x) \mid t \in T, x \in X, (t, x) \neq 1\}$$

of R is calculated as follows:

- For $t = x^i$, ($0 \leq i \leq n-2$),

$$(t, x) = tx(\overline{tx})^{-1} = x^{i+1}(\overline{x^{i+1}})^{-1} = 1.$$

- For $t = x^{n-1}$,

$$(t, x) = x^n(\overline{x^n})^{-1} = x^n.$$

Hence we obtain $X^* = \{x^n\}$. \square

Lemma 4.2. *Let M be C_n -module. Then $H^1(R, M)^{C_n} \simeq M^{C_n}$.*

Proof. Since R acts on M trivially and R is a free group with basis $\{x^n\}$, we obtain an isomorphism

$$\rho : H^1(R, M) \rightarrow M$$

defined by $\rho(f) \mapsto f(x^n)$.

Now, for any $y = x^i \in C_n$, and $f \in H^1(R, M)$, the action of y on f is given by

$$\begin{aligned} (y \cdot f)(x^n) &= yf(yx^ny^{-1}) \\ &= yf(x^i x^n x^{-i}) \\ &= yf(x^n). \end{aligned}$$

This shows that ρ is a C_n -isomorphism. Hence we have $H^1(R, M)^{C_n} \simeq M^{C_n}$. \square

Proposition 4.1. *For any C_n -module M , we have*

$$H^2(C_n, M) \simeq M^{C_n} / (1 + x + \cdots + x^{n-1})M$$

Proof. It suffices to show that the image of

$$\psi := \rho \circ \text{res} : H^1(F, M) \rightarrow M^{C_n}$$

is $(1 + x + \cdots + x^{n-1})M$. For any $[f] \in H^1(F, M)$, we have

$$\begin{aligned} \psi([f]) &= f(x^n) \\ &= (1 + x + \cdots + x^{n-1})f(x) \end{aligned}$$

where $[f]$ denotes the equivalence class of a crossed homomorphism f . This shows $\text{Im}(\psi) = (1 + x + \cdots + x^{n-1})M$. \square

We also obtain the following results. For details, see [6].

5. THE DIHEDRAL GROUP D_n

For any $n \geq 1$, let D_n be the dihedral group of order $2n$. The group D_n has a finite presentation

$$D_n = \langle \sigma, \tau \mid \sigma^n = \tau^2 = 1, \tau\sigma\tau = \sigma^{-1} \rangle.$$

Let F be the free group on $\{\sigma, \tau\}$ and R the normal closure of $\{\sigma^n, \tau^2, \tau\sigma\tau\sigma\}$ in F .

Lemma 5.1. *The group R is a free group with basis*

$$\{x, y_k, z_k \mid 0 \leq k \leq n-1\}$$

where

$$\begin{aligned} x &= \sigma^n, \\ y_0 &= \tau\sigma\tau^{-1}\sigma^{-(n-1)}, \\ y_k &= \sigma^k\tau\sigma\tau^{-1}\sigma^{-(k-1)}, \quad (1 \leq k \leq n-1), \\ z_k &= \sigma^k\tau^2\sigma^{-k} \quad (0 \leq k \leq n-1). \end{aligned}$$

Proof. It is easily seen that

$$T = \{1, \sigma, \dots, \sigma^{n-1}, \tau, \sigma\tau, \dots, \sigma^{n-1}\tau\}$$

is a Schreier transversal for R in F . Using the Reidemeister-Schreier method, we show this lemma. \square

Lemma 5.2. *Let M be any D_n -module. Then we have*

$$H^1(R, M)^{D_n} \simeq L$$

where

$$L = \left\{ (a, b, c) \in M^\sigma \oplus M^\sigma \oplus M^\tau \mid \begin{aligned} nb &= (\tau - (n-1))a, \\ (\tau - 1)a + (\tau - 1)b + (\sigma - 1)c &= 0 \end{aligned} \right\}.$$

Proposition 5.1. *For any D_n -module M , we have*

$$H^2(D_n, M) \simeq L/K$$

where

$$K = \left\{ \begin{aligned} &((1 + \sigma + \dots + \sigma^{n-1})s, \\ &(1 - \sigma^{n-1})t + (\tau - (1 + \sigma + \dots + \sigma^{n-2}))s, (1 + \tau)t) \in L \mid s, t \in M \end{aligned} \right\}.$$

6. THE GROUP $PSL(2, \mathbf{Z})$

Let $PSL(2, \mathbf{Z})$ be the projective special linear group over \mathbf{Z} . The group $PSL(2, \mathbf{Z})$ has a finite presentation

$$PSL(2, \mathbf{Z}) = \langle \sigma, \tau \mid \sigma^3 = 1, \tau^2 = 1 \rangle.$$

Let F be the free group on $\{\sigma, \tau\}$ and R the normal closure of $\{\sigma^3, \tau^2\}$ in F . To calculate a Schreier transversal for R in F , we prepare the following notations. For $m \geq 1$, $e_i \in \{1, 2\}$ ($1 \leq i \leq m$) and $k \in \{0, 1\}$, put

$$\begin{aligned} \alpha_k(e_1, \dots, e_m) &= \tau^k \sigma^{e_1} \tau \sigma^{e_2} \tau \dots \tau \sigma^{e_m}, \\ \beta_k(e_1, \dots, e_m) &= \tau^k \sigma^{e_1} \tau \sigma^{e_2} \tau \dots \tau \sigma^{e_m} \tau. \end{aligned}$$

Lemma 6.1. *Let*

$$T_1 = \left\{ \alpha_k(e_1, \dots, e_m), \beta_k(e_1, \dots, e_m) \mid k \in \{0, 1\}, m \geq 1, e_i = 1, 2 \right\},$$

and $T_2 = \{1, \tau\}$. Then $T = T_1 \cup T_2$ is a Schreier transversal for R in F .

For $m \geq 1, e_i \in \{1, 2\}$ ($1 \leq i \leq m$) and $k \in \{0, 1\}$, put

$$v = \tau^2,$$

$$w_k = \tau^k \sigma^3 \tau^{-k},$$

$$x_k(e_1, \dots, e_m) = \tau^k \sigma^{e_1} \tau \dots \tau \sigma^{e_m} \tau^2 \sigma^{-e_m} \tau^{-1} \dots \tau^{-1} \sigma^{-e_1} \tau^{-k},$$

$$y_k(e_1, \dots, e_m) = \tau^k \sigma^{e_1} \tau \dots \tau \sigma^{e_m} \tau \sigma^3 \tau^{-1} \sigma^{-e_m} \tau^{-1} \dots \tau^{-1} \sigma^{-e_1} \tau^{-k}.$$

Lemma 6.2. *The group R is a free group with basis*

$$\left\{ v, w_k, x_k(e_1, \dots, e_m), y_k(e_1, \dots, e_m) \mid k \in \{0, 1\}, m \geq 1, e_i = 1, 2 \right\}.$$

Lemma 6.3. *Let M be any $PSL(2, \mathbb{Z})$ -module. Then*

$$H^1(R, M)^{PSL(2, \mathbb{Z})} \simeq M^\tau \oplus M^\sigma.$$

Proposition 6.1. *For any $PSL(2, \mathbb{Z})$ -module M ,*

$$H^2(PSL(2, \mathbb{Z}), M) \simeq \left(M^\tau / (1 + \tau)M \right) \oplus \left(M^\sigma / (1 + \sigma + \sigma^2)M \right).$$

7. THE GROUP $SL(2, \mathbb{Z})$

Let $SL(2, \mathbb{Z})$ be the special linear group over \mathbb{Z} . The group $SL(2, \mathbb{Z})$ has a finite presentation

$$SL(2, \mathbb{Z}) = \langle \sigma, \tau \mid \sigma^3 = \tau^2, \tau^4 = 1 \rangle.$$

The elements σ and τ correspond to

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

respectively. Let F be the free group on $\{\sigma, \tau\}$ and R the normal closure of $\{\sigma^3 \tau^{-2}, \tau^4\}$ in F . To calculate a Schreier transversal for R , we prepare the following notations. For $m \geq 1, e_i \in \{1, 2\}$ ($1 \leq i \leq m$) and k ($0 \leq k \leq 3$), put

$$\alpha_k(e_1, \dots, e_m) = \tau^k \sigma^{e_1} \tau \sigma^{e_2} \tau \dots \tau \sigma^{e_m}$$

$$\beta_k(e_1, \dots, e_m) = \tau^k \sigma^{e_1} \tau \sigma^{e_2} \tau \dots \tau \sigma^{e_m} \tau$$

$$\gamma_k = \tau^k.$$

Lemma 7.1. *Let*

$$T = \bigcup_{k \in \mathbb{Z}} \left\{ \alpha_k(e_1, \dots, e_m), \beta_k(e_1, \dots, e_m), \gamma_k \mid m \geq 1, e_i = 1, 2 \right\}.$$

Then T is a Schreier transversal for R in F .

For $m \geq 1$, $e_i \in \{1, 2\}$ ($1 \leq i \leq m$) and k ($0 \leq k \leq 3$), put

$$\begin{aligned} v &= \tau^4, \\ w_0 &= \sigma^3 \tau^{-2}, \\ w_1 &= \tau \sigma^3 \tau^{-3}, \\ w_2 &= \tau^2 \sigma^3, \\ w_3 &= \tau^3 \sigma^3 \tau^{-1}, \\ x_0(e_1, \dots, e_m) &= \sigma^{e_1} \tau \dots \tau \sigma^{e_m} \tau^2 \sigma^{-e_m} \tau^{-1} \dots \tau^{-1} \sigma^{-e_1} \tau^{-2}, \\ x_1(e_1, \dots, e_m) &= \tau \sigma^{e_1} \tau \dots \tau \sigma^{e_m} \tau^2 \sigma^{-e_m} \tau^{-1} \dots \tau^{-1} \sigma^{-e_1} \tau^{-3}, \\ x_2(e_1, \dots, e_m) &= \tau^2 \sigma^{e_1} \tau \dots \tau \sigma^{e_m} \tau^2 \sigma^{-e_m} \tau^{-1} \dots \tau^{-1} \sigma^{-e_1}, \\ x_3(e_1, \dots, e_m) &= \tau^3 \sigma^{e_1} \tau \dots \tau \sigma^{e_m} \tau^2 \sigma^{-e_m} \tau^{-1} \dots \tau^{-1} \sigma^{-e_1} \tau^{-1}, \\ y_0(e_1, \dots, e_m) &= \sigma^{e_1} \tau \dots \tau \sigma^{e_m} \tau \sigma^3 \tau^{-1} \sigma^{-e_m} \tau^{-1} \dots \tau^{-1} \sigma^{-e_1} \tau^{-2}, \\ y_1(e_1, \dots, e_m) &= \tau \sigma^{e_1} \tau \dots \tau \sigma^{e_m} \tau \sigma^3 \tau^{-1} \sigma^{-e_m} \tau^{-1} \dots \tau^{-1} \sigma^{-e_1} \tau^{-3}, \\ y_2(e_1, \dots, e_m) &= \tau^2 \sigma^{e_1} \tau \dots \tau \sigma^{e_m} \tau \sigma^3 \tau^{-1} \sigma^{-e_m} \tau^{-1} \dots \tau^{-1} \sigma^{-e_1}, \\ y_3(e_1, \dots, e_m) &= \tau^3 \sigma^{e_1} \tau \dots \tau \sigma^{e_m} \tau \sigma^3 \tau^{-1} \sigma^{-e_m} \tau^{-1} \dots \tau^{-1} \sigma^{-e_1} \tau^{-1}. \end{aligned}$$

Lemma 7.2. *The group R is a free group with basis*

$$\bigcup_{0 \leq k \leq 3} \left\{ v, x_k(e_1, \dots, e_m), y_k(e_1, \dots, e_m), z_k \mid m \geq 1, e_i = 1, 2 \right\}.$$

Lemma 7.3. *Let M be any $SL(2, \mathbb{Z})$ -module. Then*

$$H^1(R, M)^{SL(2, \mathbb{Z})} \simeq N$$

where

$$N \simeq \left\{ (a, d) \in M^\tau \oplus M \mid (1 - \sigma)a = -(1 - \sigma)(1 + \sigma^3)d \right\}.$$

Proposition 7.1. *For any $SL(2, \mathbb{Z})$ -module M , we have*

$$H^2(SL(2, \mathbb{Z}), M) \simeq N/L,$$

where

$$L = \left\{ ((1 + \tau + \tau^2 + \tau^3)t, (1 + \sigma + \sigma^2)s - (1 + \tau)t) \mid s, t \in M \right\}.$$

8. THE BRAID GROUP B_3 OF INDEX THREE

Let B_3 be the braid group of index three. B_3 has a finite presentation

$$B_3 = \langle \sigma, \tau \mid \sigma^3 = \tau^2 \rangle.$$

Let F be the free group on $\{\sigma, \tau\}$ and R the normal closure of $\{\sigma^3 \tau^{-2}\}$ in F . To calculate a Schreier transversal for R , we prepare the following notations.

For $m \geq 1$, $e_i \in \{1, 2\}$ ($1 \leq i \leq m$) and $k \in \mathbb{Z}$, put

$$\begin{aligned}\alpha_k(e_1, \dots, e_m) &= \tau^k \sigma^{e_1} \tau \sigma^{e_2} \tau \dots \tau \sigma^{e_m} \\ \beta_k(e_1, \dots, e_m) &= \tau^k \sigma^{e_1} \tau \sigma^{e_2} \tau \dots \tau \sigma^{e_m} \tau \\ \gamma_k &= \tau^k.\end{aligned}$$

Lemma 8.1. *Let*

$$T = \bigcup_{k \in \mathbb{Z}} \left\{ \alpha_k(e_1, \dots, e_m), \beta_k(e_1, \dots, e_m), \gamma_k \mid m \geq 1, e_i = 1, 2 \right\}.$$

Then T is a Schreier transversal for R in F .

For $m \geq 1$, $e_i \in \{1, 2\}$ ($1 \leq i \leq m$) and $k \in \mathbb{Z}$, put

$$\begin{aligned}x_k(e_1, \dots, e_m) &= \tau^k \sigma^{e_1} \tau \dots \tau \sigma^{e_m} \tau^2 \sigma^{-e_m} \tau^{-1} \dots \tau^{-1} \sigma^{-e_1} \tau^{-(k+2)}, \\ y_k(e_1, \dots, e_m) &= \tau^k \sigma^{e_1} \tau \dots \tau \sigma^{e_m} \tau \sigma^3 \tau^{-1} \sigma^{-e_m} \tau^{-1} \dots \tau^{-1} \sigma^{-e_1} \tau^{-(k+2)}, \\ z_k &= \tau^k \sigma^3 \tau^{-(k+2)}.\end{aligned}$$

Lemma 8.2. *The group R is a free group with basis*

$$\bigcup_{k \in \mathbb{Z}} \left\{ x_k(e_1, \dots, e_m), y_k(e_1, \dots, e_m), z_k \mid m \geq 1, e_i = 1, 2 \right\}.$$

Lemma 8.3. *Let M be any B_3 -module. Then*

$$H^1(R, M)^{B_3} \simeq M.$$

Proposition 8.1. *For any B_3 -module M , we have*

$$H^2(B_3, M) \simeq M / (1 + \sigma + \sigma^2)M + (1 + \tau)M.$$

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